$$\alpha = \frac{\tau}{I} = \frac{375.0 \text{ N-m}}{84.38 \text{ kg-m}^2} = 4.44 \frac{\text{rad}}{\text{s}^2}.$$

Significance

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case.

10.7 **Check Your Understanding** The fan blades on a jet engine have a moment of inertia 30.0 kg-m^2. In 10 s, they rotate counterclockwise from rest up to a rotation rate of 20 rev/s. (a) What torque must be applied to the blades to achieve this angular acceleration? (b) What is the torque required to bring the fan blades rotating at 20 rev/s to a rest in 20 s?

10.8 | Work and Power for Rotational Motion

Learning Objectives

By the end of this section, you will be able to:

- Use the work-energy theorem to analyze rotation to find the work done on a system when it is rotated about a fixed axis for a finite angular displacement
- Solve for the angular velocity of a rotating rigid body using the work-energy theorem
- Find the power delivered to a rotating rigid body given the applied torque and angular velocity
- Summarize the rotational variables and equations and relate them to their translational counterparts

Thus far in the chapter, we have extensively addressed kinematics and dynamics for rotating rigid bodies around a fixed axis. In this final section, we define work and power within the context of rotation about a fixed axis, which has applications to both physics and engineering. The discussion of work and power makes our treatment of rotational motion almost complete, with the exception of rolling motion and angular momentum, which are discussed in **Angular Momentum**. We begin this section with a treatment of the work-energy theorem for rotation.

Work for Rotational Motion

Now that we have determined how to calculate kinetic energy for rotating rigid bodies, we can proceed with a discussion of the work done on a rigid body rotating about a fixed axis. **Figure 10.39** shows a rigid body that has rotated through an angle $d\theta$ from A to B while under the influence of a force $\vec{\mathbf{F}}$. The external force $\vec{\mathbf{F}}$ is applied to point P, whose position is $\vec{\mathbf{r}}$, and the rigid body is constrained to rotate about a fixed axis that is perpendicular to the page and passes through O. The rotational axis is fixed, so the vector $\vec{\mathbf{r}}$ moves in a circle of radius r, and the vector $d\vec{\mathbf{s}}$ is perpendicular to $\vec{\mathbf{r}}$.

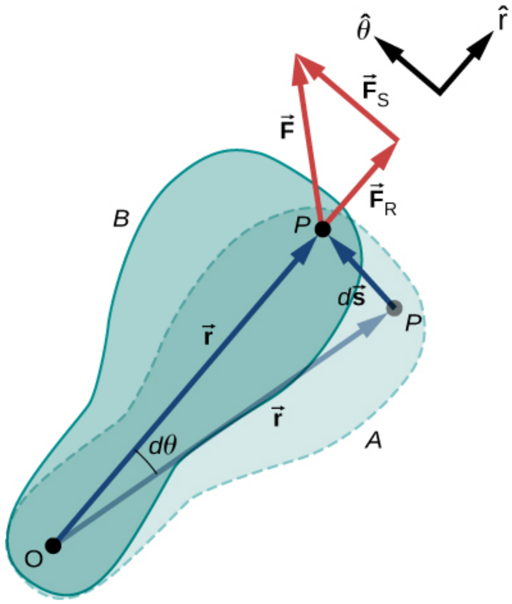

Figure 10.39 A rigid body rotates through an angle $d\theta$ from A to B by the action of an external force $\vec{\mathbf{F}}$ applied to point P.

From **Equation 10.2**, we have

$$\vec{\mathbf{s}} = \vec{\boldsymbol{\theta}} \times \vec{\mathbf{r}}.$$

Thus,

$$d\vec{\mathbf{s}} = d(\vec{\boldsymbol{\theta}} \times \vec{\mathbf{r}}) = d\vec{\boldsymbol{\theta}} \times \vec{\mathbf{r}} + d\vec{\mathbf{r}} \times \vec{\boldsymbol{\theta}} = d\vec{\boldsymbol{\theta}} \times \vec{\mathbf{r}}.$$

Note that $d\vec{\mathbf{r}}$ is zero because $\vec{\mathbf{r}}$ is fixed on the rigid body from the origin O to point P. Using the definition of work, we obtain

$$W = \int \sum \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int \sum \vec{\mathbf{F}} \cdot (d\vec{\boldsymbol{\theta}} \times \vec{\mathbf{r}}) = \int d\vec{\boldsymbol{\theta}} \cdot (\vec{\mathbf{r}} \times \sum \vec{\mathbf{F}})$$

where we used the identity $\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{a}})$. Noting that $(\vec{\mathbf{r}} \times \sum \vec{F}) = \sum \vec{\boldsymbol{\tau}}$, we arrive at the expression for the **rotational work** done on a rigid body:

$$W = \int \sum \vec{\boldsymbol{\tau}} \cdot d\vec{\boldsymbol{\theta}}. \tag{10.27}$$

The total work done on a rigid body is the sum of the torques integrated over the angle through which the body rotates. The incremental work is

$$dW = \left(\sum_i \tau_i\right) d\theta \tag{10.28}$$

where we have taken the dot product in **Equation 10.27**, leaving only torques along the axis of rotation. In a rigid body, all particles rotate through the same angle; thus the work of every external force is equal to the torque times the common incremental angle $d\theta$. The quantity $\left(\sum_i \tau_i\right)$ is the net torque on the body due to external forces.

Similarly, we found the kinetic energy of a rigid body rotating around a fixed axis by summing the kinetic energy of each particle that makes up the rigid body. Since the work-energy theorem $W_i = \Delta K_i$ is valid for each particle, it is valid for the sum of the particles and the entire body.

Work-Energy Theorem for Rotation

The work-energy theorem for a rigid body rotating around a fixed axis is

$$W_{AB} = K_B - K_A \tag{10.29}$$

where

$$K = \frac{1}{2}I\omega^2$$

and the rotational work done by a net force rotating a body from point A to point B is

$$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i \right) d\theta. \tag{10.30}$$

We give a strategy for using this equation when analyzing rotational motion.

Problem-Solving Strategy: Work-Energy Theorem for Rotational Motion

1. Identify the forces on the body and draw a free-body diagram. Calculate the torque for each force.
2. Calculate the work done during the body's rotation by every torque.
3. Apply the work-energy theorem by equating the net work done on the body to the change in rotational kinetic energy.

Let's look at two examples and use the work-energy theorem to analyze rotational motion.

Example 10.17

Rotational Work and Energy

A $12.0\,\text{N}\cdot\text{m}$ torque is applied to a flywheel that rotates about a fixed axis and has a moment of inertia of $30.0\,\text{kg}\cdot\text{m}^2$. If the flywheel is initially at rest, what is its angular velocity after it has turned through eight revolutions?

Strategy

We apply the work-energy theorem. We know from the problem description what the torque is and the angular displacement of the flywheel. Then we can solve for the final angular velocity.

Solution

The flywheel turns through eight revolutions, which is 16π radians. The work done by the torque, which is constant and therefore can come outside the integral in **Equation 10.30**, is

$$W_{AB} = \tau(\theta_B - \theta_A).$$

We apply the work-energy theorem:

$$W_{AB} = \tau(\theta_B - \theta_A) = \frac{1}{2}I\omega_B^2 - \frac{1}{2}I\omega_A^2.$$

With $\tau = 12.0\,\text{N}\cdot\text{m}$, $\theta_B - \theta_A = 16.0\pi\,\text{rad}$, $I = 30.0\,\text{kg}\cdot\text{m}^2$, and $\omega_A = 0$, we have

$$12.0\,\text{N-m}(16.0\pi\,\text{rad}) = \frac{1}{2}(30.0\,\text{kg}\cdot\text{m}^2)(\omega_B^2) - 0.$$

Therefore,

$$\omega_B = 6.3 \text{ rad/s}.$$

This is the angular velocity of the flywheel after eight revolutions.

Significance

The work-energy theorem provides an efficient way to analyze rotational motion, connecting torque with rotational kinetic energy.

Example 10.18

Rotational Work: A Pulley

A string wrapped around the pulley in Figure 10.40 is pulled with a constant downward force $\vec{\mathbf{F}}$ of magnitude 50 N. The radius R and moment of inertia I of the pulley are 0.10 m and $2.5 \times 10^{-3} \text{kg-m}^2$, respectively. If the string does not slip, what is the angular velocity of the pulley after 1.0 m of string has unwound? Assume the pulley starts from rest.

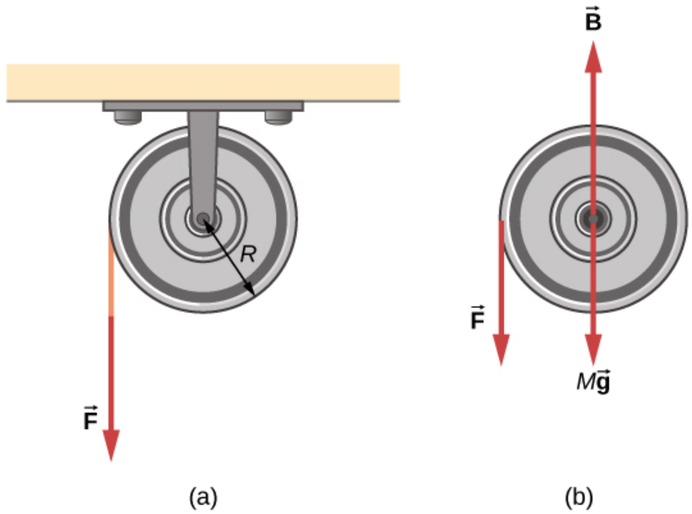

Figure 10.40 (a) A string is wrapped around a pulley of radius R. (b) The free-body diagram.

Strategy

Looking at the free-body diagram, we see that neither $\vec{\mathbf{B}}$, the force on the bearings of the pulley, nor $M\vec{\mathbf{g}}$, the weight of the pulley, exerts a torque around the rotational axis, and therefore does no work on the pulley. As the pulley rotates through an angle θ, $\vec{\mathbf{F}}$ acts through a distance d such that $d = R\theta$.

Solution

Since the torque due to $\vec{\mathbf{F}}$ has magnitude $\tau = RF$, we have

$$W = \tau\theta = (FR)\theta = Fd.$$

If the force on the string acts through a distance of 1.0 m, we have, from the work-energy theorem,

$$\begin{aligned} W_{AB} &= K_B - K_A \\ Fd &= \frac{1}{2}I\omega^2 - 0 \\ (50.0\,\text{N})(1.0\,\text{m}) &= \frac{1}{2}(2.5 \times 10^{-3}\,\text{kg-m}^2)\omega^2. \end{aligned}$$

Solving for ω, we obtain

$$\omega = 200.0\,\text{rad/s}.$$

Power for Rotational Motion

Power always comes up in the discussion of applications in engineering and physics. Power for rotational motion is equally as important as power in linear motion and can be derived in a similar way as in linear motion when the force is a constant. The linear power when the force is a constant is $P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$. If the net torque is constant over the angular displacement, **Equation 10.25** simplifies and the net torque can be taken out of the integral. In the following discussion, we assume the net torque is constant. We can apply the definition of power derived in **Power** to rotational motion. From **Work and Kinetic Energy**, the instantaneous power (or just power) is defined as the rate of doing work,

$$P = \frac{dW}{dt}.$$

If we have a constant net torque, **Equation 10.25** becomes $W = \tau\theta$ and the power is

$$P = \frac{dW}{dt} = \frac{d}{dt}(\tau\theta) = \tau\frac{d\theta}{dt}$$

or

$$P = \tau\omega. \tag{10.31}$$

Example 10.19

Torque on a Boat Propeller

A boat engine operating at $9.0 \times 10^4\ \text{W}$ is running at 300 rev/min. What is the torque on the propeller shaft?

Strategy

We are given the rotation rate in rev/min and the power consumption, so we can easily calculate the torque.

Solution

$$300.0\,\text{rev/min} = 31.4\,\text{rad/s};$$

$$\tau = \frac{P}{\omega} = \frac{9.0 \times 10^4\,\text{N} \cdot \text{m/s}}{31.4\,\text{rad/s}} = 2864.8\,\text{N} \cdot \text{m}.$$

Significance

It is important to note the radian is a dimensionless unit because its definition is the ratio of two lengths. It therefore does not appear in the solution.

10.8 Check Your Understanding A constant torque of $500\,\text{kN} \cdot \text{m}$ is applied to a wind turbine to keep it rotating at 6 rad/s. What is the power required to keep the turbine rotating?

Rotational and Translational Relationships Summarized

The rotational quantities and their linear analog are summarized in three tables. **Table 10.5** summarizes the rotational variables for circular motion about a fixed axis with their linear analogs and the connecting equation, except for the centripetal acceleration, which stands by itself. **Table 10.6** summarizes the rotational and translational kinematic equations. **Table 10.7** summarizes the rotational dynamics equations with their linear analogs.

Rotational	Translational	Relationship
θ	x	$\theta = \frac{s}{r}$
ω	v_t	$\omega = \frac{v_t}{r}$
α	a_t	$\alpha = \frac{a_t}{r}$
	a_c	$a_c = \frac{v_t^2}{r}$

Table 10.5 Rotational and Translational Variables: Summary

Rotational	Translational
$\theta_f = \theta_0 + \bar{\omega} t$	$x = x_0 + \bar{v} t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x_f = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega^2{}_0 + 2\alpha(\Delta\theta)$	$v_f^2 = v^2{}_0 + 2a(\Delta x)$

Table 10.6 Rotational and Translational Kinematic Equations: Summary

Rotational	Translational
$I = \sum_i m_i r_i^2$	m
$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$
$\sum_i \tau_i = I\alpha$	$\sum_i \vec{\mathbf{F}}_i = m\vec{\mathbf{a}}$
$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right) d\theta$	$W = \int \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$
$P = \tau\omega$	$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$

Table 10.7 Rotational and Translational Equations: Dynamics